How to estimate the radius of convergence of your solution without solving the ODE.

<u>Theorem</u>: For the ODE

$$y'' + p(x)y' + q(x)y = 0$$

if $p(x)$, $q(x)$ are analytic in a region including x_0, then the series solution about x_0 is also analytic within this region.

$f(x)$ analytic at a point: $f(x)$ admits a Taylor series expansion about x_0.

Equivalently, if $f(x)$, regarded as a complex function, is differentiable.

<u>Example</u>: $e^{-\frac{1}{x}}$ is infinitely differentiable but not analytic at $x=0$. In fact, $\frac{d^n}{dx^n}(e^{-\frac{1}{x}})\Big|_{x_0=0} = 0$

<u>Corollary</u>: If $p(x)$, $q(x)$ has singularities $z_1, z_2, \cdots, z_k$ over $\mathbb{C}$, then for the series solution about x_0, the radius of convergence is at least $\min\{\text{distance between } x_0 \text{ and } z_i\}_{i=1,\cdots,k}$

<u>Examples</u> 1. $y'' - xy' + y = 0$. $x_0 = 0$

$p(x) = -x$, $q(x) = 1$, both analytic. No singularity.

Radius of convergence $= \infty$.

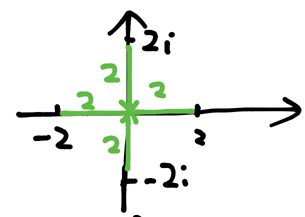
Example 2. $(x^4-16)y'' + xy' - e^x y = 0 \quad x_0 = 0,\ x_0 = 4$

$$y'' + \frac{x}{x^4-16} y' - \frac{e^x}{x^4-16} y = 0.$$

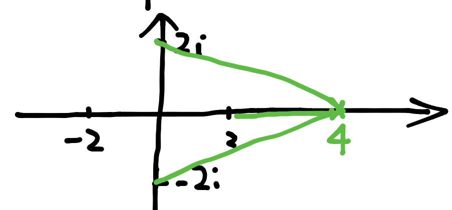
$p(x) = \dfrac{x}{x^4-16}$, $q(x) = \dfrac{-e^x}{x^4-16}$ not defined when $x^4 - 16 = 0$.

$x^4 - 16 = 0 \Rightarrow (x^2-4)(x^2+4) = 0 \Rightarrow x = -2, 2, -2i, 2i$

The series sol'n about $x_0 = 0$ has radius of conv. ≥ 2



The series sol'n about $x_0 = 4$ has radius of convergence ≥ 2



Why we need this estimate: In engineering practice, to achieve enough precision, we only need first few terms of the series sol'n. But to obtain radius of convergence just as in Calc 2, we need to know all the terms. This usually means much more computation. This estimate saves us from such computations.

More examples: $y'' - xy' + y = 0$, $x_0 = 1$

Step 1: $y = \sum_{n=0}^{\infty} a_n (x-1)^n$, $y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$, $y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$

Step 2: $\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \boxed{x} \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^n$

Trick: $x = (x-1) + 1$. (Divide x by $x-1$. quotient $= 1$, remainder $= 1$)

Multiplication

Second term becomes $-(x-1)\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$$= -\sum_{n=1}^{\infty} n a_n (x-1)^{n} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

We have to deal four series

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - \sum_{n=1}^{\infty} n a_n (x-1)^{n} - \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n}$$

Unify exponents

$m = n-2$, $n = m+2$; $m = n$, $n = m$; $m = n-1$, $n = m+1$; $m = n$, $n = m$

$$\sum_{m+2=2 \Rightarrow m=0} (m+2)(m+1) a_{m+2} (x-1)^m - \sum_{m=1}^{\infty} m a_m (x-1)^m$$

$$- \sum_{m+1=1 \Rightarrow m=0}^{\infty} (m+1) a_{m+1} (x-1)^m + \sum_{m=0}^{\infty} a_m (x-1)^m.$$

Unify the sum indices

$$2\cdot 1\cdot a_2 (x-1)^0 + \sum_{m=1}^{\infty} (m+2)(m+1) a_{m+2} (x-1)^m \quad \text{0th term out}$$

$$- \sum_{m=1}^{\infty} m a_m (x-1)^m \quad \text{keep it}$$

$$- 1\cdot a_1 \cdot (x-1)^0 - \sum_{m=1}^{\infty} (m+1) a_{m+1} (x-1)^m \quad \text{0th term out}$$

$$+ a_0 (x-1)^0 + \sum_{m=1}^{\infty} a_m (x-1)^m. \quad \text{0th term out}$$

$$(2a_2 - a_1 + a_0) + \sum_{m=1}^{\infty} \left[(m+1)(m+2) a_{m+2} - m a_m - (m+1) a_{m+1} + a_m \right] (x-1)^m = 0$$

Step 3: $\begin{cases} 2a_2 - a_1 + a_0 = 0 \\ (m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m-1)a_m = 0 \qquad m \geq 1 \end{cases}$

Set a_0, a_1 arbitrary constants,

$$a_2 = \frac{1}{2}a_1 - \frac{1}{2}a_0 \qquad \text{from first eqn}$$

$$a_{m+2} = \frac{a_{m+1}}{m+2} - \frac{m-1}{(m+2)(m+1)}a_m \qquad \text{from second eqn}$$

$m=1$ $\quad a_3 = \frac{a_2}{3} - 0 \cdot a_1 = \frac{1}{3}a_2 = \frac{1}{6}a_1 - \frac{1}{6}a_0$

$m=2$ $\quad a_4 = \frac{a_3}{4} - \frac{1}{4\cdot 3}a_2 = \frac{1}{24}a_1 - \frac{1}{24}a_0 - \frac{1}{24}a_1 + \frac{1}{24}a_0$

$$= 0$$

$m=3$ $\quad a_5 = \frac{a_4}{5} - \frac{2}{5\cdot 4}a_3 = -\frac{1}{60}a_1 + \frac{1}{60}a_0$

Step 4: $y = a_0 + a_1(x-1) + \left(\frac{1}{2}a_1 - \frac{1}{2}a_0\right)(x-1)^2 + \left(\frac{1}{6}a_1 - \frac{1}{6}a_0\right)(x-1)^3$

$$+ \left(-\frac{1}{60}a_1 + \frac{1}{60}a_0\right)(x-1)^5 + \cdots$$

$$= a_0\left(1 - \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{60}(x-1)^5 + \cdots\right)$$

$$+ a_1\left((x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{1}{60}(x-1)^5 + \cdots\right)$$

Radius of convergence? $\quad p(x) = -x$, $q(x) = 1$ analytic everywhere

$$\text{RoC} = \infty.$$

In general, if RoC $= R$, then the series sol'n makes sense within the region $|x-x_0|<R$. (domain of the solution)

Example: $(2+x^2)y''-xy'+4y=0 \quad x_0=0$

Step 1: $y=\sum_{n=0}^{\infty} a_n x^n$

Step 2 $\Rightarrow$

$$(4a_2+4a_0)+(12a_3+3a_1)x + \sum_{m=2}^{\infty}\left[2(m+2)(m+1)a_{m+2}+m(m-1)a_m-ma_m+4a_m\right]x^m=0$$

Step 3:
$$\begin{cases} 4a_2+4a_0=0 \\ 12a_3+3a_1=0 \\ 2(m+2)(m+1)a_{m+2}+(m^2-2m+4)a_m=0 \end{cases}$$

Step 4: $y=a_0\left(1-x^2-\frac{1}{6}x^4-\frac{1}{30}x^6+\cdots\right) + a_1\left(x-\frac{1}{4}x^3+\frac{7}{160}x^5-\frac{19}{1920}x^7+\cdots\right)$

Radius of Convergence $\geq \sqrt{2}$, b/c $\sqrt{2}i, -\sqrt{2}i$ singular pts.

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